Table 3 Maximum stresses and deflection in 7-ply laminate

S	$\bar{\sigma}_x$ $(a/2, a/2, \pm 1/2)$	$\bar{\sigma}_y$ $(a/2, a/2, \pm 3/8)$	$\bar{\tau}_{xz}$ $(0, a/2, 0)$	$\bar{\tau}_{yz}$ $(a/2, 0, 0)$	$\bar{\tau}_{xy}$ $(0, 0, \pm 1/2)$	$\bar{w}$ $(a/2, a/2, 0)$
			Elasticity			
2	1.284	1.039	0.178	0.238	−0.0775	12.342
	−0.880	−0.838	0.229(0.16)	0.239(0.02)	0.0579	
4	0.679	0.623	0.219	0.236	−0.0356	4.153
	−0.645	−0.610	0.223(0.12)		0.0347	
10	±0.548	0.457	0.255	0.219	−0.0237	1.529
		−0.458	0.255(−0.02)		0.0238	
20	±0.539	0.419	0.267	0.210	∓0.0219	1.133
		−0.420				
50	±0.539	±0.407	0.271	0.206	∓0.0214	1.021
100	±0.539	±0.405	0.272	0.205	∓0.0213	1.005
			CPT			
	±0.539	±0.404	0.272	0.205	∓0.0213	1

Table 4 Maximum stresses and deflection in 9-ply laminate

S	$\bar{\sigma}_x$ $(a/2, a/2, \pm 1/2)$	$\bar{\sigma}_y$ $(a/2, a/2, \pm 2/5)$	$\bar{\tau}_{xz}$ $(0, a/2, 0)$	$\bar{\tau}_{yz}$ $(a/2, 0, 0)$	$\bar{\tau}_{xy}$ $(0, 0, \pm 1/2)$	$\bar{w}$ $(a/2, a/2, 0)$
			Elasticity			
2	1.260	1.051	0.204	0.194	−0.0722	12.288
	−0.866	−0.824	0.224(0.23)	0.211(−0.10)	0.0534	
4	0.684	0.628	0.223	0.223	−0.0337	4.079
	−0.649	−0.612	0.223(0.01)	0.225(−0.06)	0.0328	
10	±0.551	±0.477	0.247	0.226	−0.0233	1.512
				0.226(−0.01)	0.0235	
20	±0.541	±0.444	0.255	0.221	∓0.0218	1.129
50	±0.539	±0.433	0.258	0.219	∓0.0214	1.021
100	±0.539	±0.431	0.259	0.219	∓0.0213	1.005
			CPT			
	±0.539	±0.431	0.259	0.219	∓0.0213	1

ratio $S = 10$, the maximum stresses are all predicted to within 10% accuracy by CPT in the 9-layer laminate. However, the exact deflection in this case is still 51% higher than the CPT result. Thus, as in previous studies, the stresses generally converge to the CPT values faster than the deflection.

Curves showing the thickness distribution of stresses $\bar{\sigma}_x$, $\bar{\tau}_{xz}$ and displacement $\bar{u}$ along the vertical lines, $x = \text{const}$, $y = \text{const}$, on which each of the functions assumes its maximum value, are given in Figs. 1–3 for the 9-ply laminate. The exact results are indicated for three different aspect ratios, $S = 2$, 4, 10 and the CPT results (independent of S) are also shown. To avoid loss of detail in the curves of Fig. 1, it is necessary to utilize the inserts representing the behavior of $\bar{\sigma}_x$ near $\bar{z} = \pm\frac{1}{2}$ for the case $S = 2$. Rapid convergence of the exact functions to CPT can be observed despite the characteristic zig-zag appearance of the deformed normal as shown in Fig. 3.

In conclusion, the preceding results suggest that a conservative estimate of the magnitude of the error reflected in the simplifying assumptions of CPT for multilayered systems can be achieved by comparison of exact and approximate solutions for laminates consisting of only several layers.

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A Laser Velocimeter for Reynolds Stress and Other Turbulence Measurements

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Nomenclature

c = speed of light

$\hat{e}_x, \hat{e}_y$ = unit vectors in the x and y directions

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$\hat{e}_1, \hat{e}_2$ = unit vectors along incident intersecting beams
U_o = maximum velocity in the stream (x) direction
u = $\bar{u}+u'$, velocity in x direction
v = $\bar{v}+v'$, velocity in y direction
$\langle u'v' \rangle$ = Reynolds stress
$\vec{w}$ = $u\hat{e}_x + v\hat{e}_y$, total velocity vector
$\vec{w}'$ = $u'\hat{e}_x + v'\hat{e}_y$, instantaneous vector fluctuation
λ_o = wavelength of laser light
ν_D = Doppler frequency
$\Delta\nu_D$ = spectral half-width of Doppler signal
ν_M = frequency shift introduced into one of incident beams
σ^2 = $\langle \Delta\nu_D{}^2 \rangle / \langle \nu_D \rangle^2$, variance of Doppler signal
θ = angle between incident intersection beams
ϕ = angle between plane of beams and mean flow direction
$'$ = fluctuation quantity
$\langle \ \rangle$ = time average

Introduction

MEASUREMENT of turbulent fluctuations in the streamwise direction by the laser Doppler technique is well established, having been introduced by Goldstein and Hagan in 1967,[1] and utilized by many others in recent years. However, in spite of much developmental work there appear to be yet-undemonstrated important applications of the powerful technique. One such example is measurement of the Reynolds stress, $\langle u'v' \rangle$, of great concern in turbulence studies. Although the desirability of measuring Reynolds stress was first mentioned by Goldstein and Hagan, no method was suggested, and apparently there has not yet been a demonstration of such a measurement by laser Doppler in the literature. A second important application involves measurement of turbulence fluctuations in a given direction which are greater than the mean flow velocity in that direction, such as turbulence normal to the mean flow. In such cases, directional ambiguity and low frequency noise of ordinary laser Doppler instruments prohibit turbulence measurements.

A simple laser velocimeter for turbulence measurements including Reynolds stress and normal fluctuations has been developed in the Karman Laboratory of Fluid Mechanics and used in a pipe flow measure $\langle u \rangle$, $\langle u'^2 \rangle$, $\langle v'^2 \rangle$ and $\langle u'v' \rangle$ as functions of distance from the wall.

Theory and Background of the New Techniques

1. *General*

The theory of the laser Doppler technique and the interpretation of the signals obtained has been extensively given elsewhere.[2] Radiation scattered from the intersection point of two coherent beams from the same laser source is mixed on a photodetector to yield a beating or Doppler frequency proportional to the local velocity at that point and of magnitude

$$\nu_D = (n\vec{w}/\lambda_o)\cdot(\hat{e}_1 - \hat{e}_2) \tag{1}$$

Only one velocity component is measured, lying in the plane of the two intersecting beams and perpendicular to their bisector. Three commonly used laser Doppler systems are the "cross reference beam" system introduced by Goldstein[3] (one beam used as a local oscillator), the "dual scatter" technique developed by Brayton[4] (only scattered light from both beams detected) and the "Dopplermeter" developed by Rudd[5] (all forward scattered and unscattered radiation collected). These systems are equivalent in principle and each has applications to which it is better suited than the others.

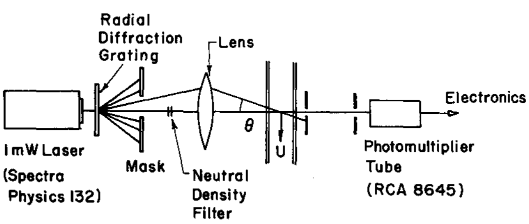

Fig. 1 Apparatus schematic.

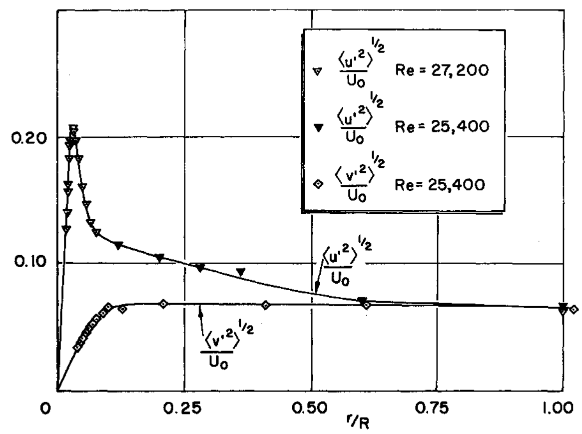

Fig. 2 Measurement of axial and radial turbulence across pipe.

For turbulent flow, it follows from Eq. (1) that

$$\langle \Delta\nu_D{}^2 \rangle = (n/\lambda_o)^2 \langle [\vec{w}'\cdot(\hat{e}_1 - \hat{e}_2)]^2 \rangle \tag{2}$$

where it has been assumed that time averages of fluctuations are zero. Now, if ϕ is the angle between the plane of the intersecting beams and the x axis and the instantaneous fluctuation velocity vector is $\vec{w}' = u'\hat{e}_x + v'\hat{e}_y$, then Eq. (2) becomes

$$\langle \Delta\nu_D{}^2 \rangle = 4(n/\lambda_o)^2 \sin^2\theta/2 \times [\langle u'^2 \rangle \cos^2\phi + 2\langle u'v' \rangle \cos\phi \sin\phi + \langle v'^2 \rangle \sin^2\phi] \tag{3}$$

since

$$(\hat{e}_1 - \hat{e}_2)\cdot\hat{e}_x = 2\sin(\theta/2)\cos\phi, \text{ etc.}$$

2. *Measurement of turbulence intensities greater than the local mean flow*

The usual turbulence measurement made with laser Doppler velocimeters is in the direction of mean flow, where typically $\langle \nu_D \rangle \gg \langle \Delta\nu^2 \rangle^{1/2}$. However, when the mean flow becomes small or zero $\langle \nu_D \rangle$ can become of the order of or smaller than the spectral broadening, and the signal becomes uninterpretable.

This Note demonstrates the feasibility of obtaining such turbulence measurements by biasing the Doppler signal at some center frequency ν_M selected to be much larger than the turbulent broadening of the signal. This is accomplished by frequency shifting one of the incident beams an amount ν_M so that Eq. (1) becomes

$$\nu_D = \nu_M + (n\vec{w}/\lambda_o)\cdot(\hat{e}_1 - \hat{e}_2) + 0(\nu_M w/c)$$

The last term is negligible since typically $(w/c) \approx 10^{-8}$ and Eq. (3) is still valid. Such frequency shifting can be accomplished in various ways[6-8] and for the present work a rotating radial diffraction grating from an optical encoder has been conveniently employed, similar to one first demonstrated by Stevenson.[7]

3. *Measurement of the Reynolds stress*

It is evident from Eq. (3) that the Reynolds stress $\langle u'v' \rangle$ may be obtained directly from the difference of the variances measured in two perpendicular directions, $\phi_{1,2} = \pm 45°$, with respect to the mean flow direction, $\phi = 0$.† Using Eq. (3),

$$\sigma_{1,2}{}^2 = [\langle u'^2 \rangle + \langle v'^2 \rangle \pm 2\langle u'v' \rangle]/2U_o{}^2$$

so that

$$\sigma_1{}^2 + \sigma_2{}^2 = [\langle u'^2 \rangle + \langle v'^2 \rangle]/U_o{}^2 \tag{4}$$

$$\sigma_1{}^2 - \sigma_2{}^2 = 2\langle u'v' \rangle/U_o{}^2 \tag{5}$$

Thus, only one receiver is required to measure $\langle u'v' \rangle$ and

† This was pointed out to the author by H. W. Liepmann in a discussion on the well-established crossed hot-wire method of $\langle u'v' \rangle$ measurement.

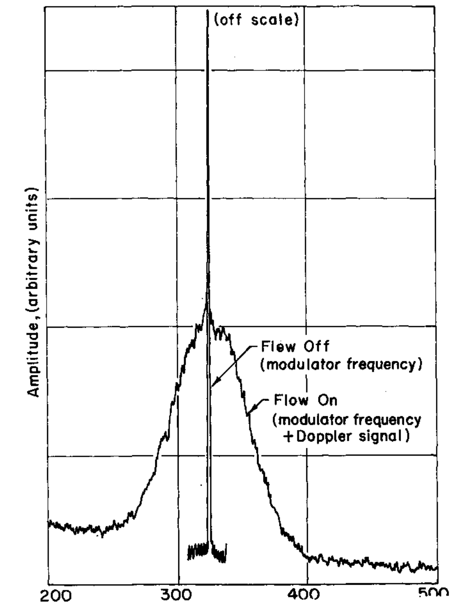

Fig. 3 Typical radial turbulence signal, $v_M = 325$ kHz.

simultaneous measurement of u' and v', multiplication, and averaging, is not necessary.

Apparatus and results: The newly developed velocimeter can measure mean flow components and turbulence quantities $\langle u'^2 \rangle$, $\langle v'^2 \rangle$, and $\langle u'v' \rangle$ with minimal adjustment of only one optical component, a radial diffraction grating, serving as a combination beam splitter, measurement direction selector, and frequency modulator. The apparatus is diagramed in Fig. 1. Two of the diffraction orders are selected by a mask and focused in the test section with a single converging lens guaranteeing "self alignment," with beams focusing at the same point. Although there are many possibilities, it has been convenient to use a "reference mode" system with the zero-order (undeflected) beam serving as the reference beam so that alignment through the fixed lens, apertures, and photomultiplier is independent of translation of the grating.

Different directions of measurement (various ϕ) are selected by translating the diffraction grating parallel to itself, and locating the incident laser beam at different positions around the circumference of the wheel at constant radius.

Most measurements are made with the grating stationary, although by rotating the grating any turbulent spectrum can be shifted in frequency by 84.2 Hz/rpm. This biasing is crucial to obtain $\langle v'^2 \rangle$ ($\phi = 90°$) and also useful to bring other Doppler signals into more convenient frequency ranges.

Representative turbulence measurements made in a $\frac{1}{2}$ inch-square lucite pipe ($l/d = 65$) are presented. Axial and radial fluctuations across the pipe are shown in Fig. 2. Turbulent broadening about the biasing frequency is shown in Fig. 3 in a typical determination of $\langle v'^2 \rangle$. Figure 4 presents measurements of $\langle u'v' \rangle$ across the pipe and the corresponding turbulent energy from Eq. (4). Secondary (radial) flow was also measured, at maximum about 0.5% of the centerline axial velocity.

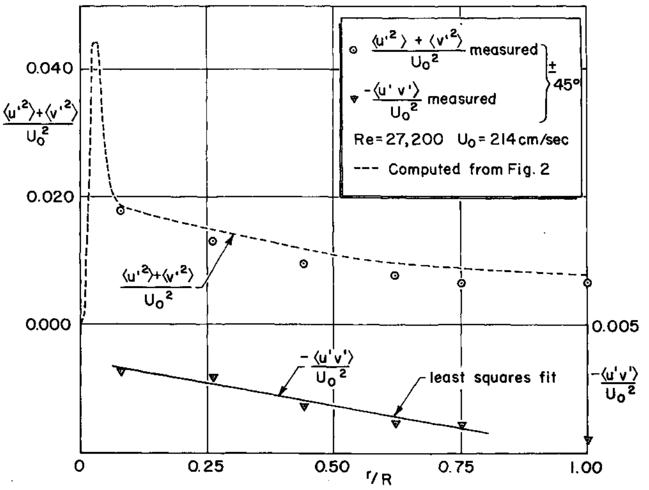

Fig. 4 Reynolds stress and turbulence energy across pipe.

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Buckling of a Circular Cylindrical Shell in Axial Compression and SS4 Boundary Conditions

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THE influence of boundary conditions on the buckling loads of circular cylindrical shells is well known and has received much attention in literature. Considering the buckling of a finite cylinder in axial compression, one finds that for most boundary conditions only numerical data is available. Closed form solutions exist only for the classical SS3 boundary condition, e.g.,[1] and for the SS1 boundary condition.[2] For a semi-infinite cylinder there exists a closed form solution also for the SS2 boundary condition.[3] The purpose of this Note is to give closed form bounds to the buckling problem of a circular cylindrical shell in axial compression for the SS4 boundary conditions. Although the solution is approximate, it is shown that the true solution is bounded between two very close quantities.

The Donnell equations,[4] for axial compression, are

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